Int. J. Solids Structures, 1968, Vol. 4, pp. 875 to 883. Pergamon Press. Printed in Great Britain

ON A VARIATIONAL THEOREM FOR INCOMPRESSIBLE AND NEARLY-INCOMPRESSIBLE ORTHOTROPIC ELASTICITY

ROBERT L. TAYLOR and KARL S. PISTER
University of California, Berkeley, California

LEONARD R. HERRMANN
University of California, Davis, California

Abstract—A mixed variational theorem for linear orthotropic thermoelastic solids is presented. The mechanical state variables are taken to be the displacement vector and a scalar stress variable. The Euler equations of the variational principle are the displacement equations of equilibrium and a condition relating the stress variable to strain and temperature change. An important feature of the principle is that the field equations for both compressible and incompressible solids may be generated. In connection with applications to the development of finite element computer algorithms for the solution of boundary value problems a well-conditioned system of equations is obtained for nearly-incompressible solids.

INTRODUCTION

THE solution of boundary value problems in linear, isotropic elastostatics is often most fruitfully accomplished by solving an equivalent variational problem. This technique has been employed extensively in recent years in the development of computer algorithms based upon a finite element Ritz method. However, applications in such fields as structural integrity analysis for solid rocket motors have encountered difficulty as a result of the near-incompressibility of typical solid propellants. The obstacle arises from the existence of a singularity in the displacement equations of equilibrium for an incompressible isotropic elastic solid, i.e. one for which Poisson's ratio $\nu = 0{\cdot}5$. Furthermore, for slightly compressible materials ($0{\cdot}49 < \nu < 0{\cdot}50$) the variational problem does not lead to a numerically well-behaved system of equations. For example, experience has shown that the Theorem of Minimum Potential Energy cannot be applied to problems in which ν falls within the limits above. This difficulty can be traced to numerical instability in the inversion of the strain–stress constitutive equation (the determinant of the elastic compliance matrix vanishes for an incompressible solid) in the process of forming the strain energy density. With this motivation Herrmann and Toms [1] reformulated the constitutive equations for a linear, isotropic thermoelastic solid and later Herrmann [2] exhibited a variational principle valid for all admissible values of Poisson's ratio including $\nu = 0{\cdot}5$. Concomitantly, the Ritz method of finite element numerical analysis associated with the new mixed variational principle was shown to be substantially superior to the minimum potential energy formulation [2].

A similar problem has arisen in dealing with filament-reinforced solids. For such composite materials in which often a nearly-incompressible matrix is combined with elastic reinforcement, the Ritz method based upon the Minimum Potential Energy Theorem leads to unsatisfactory results from a computational standpoint. Furthermore, the limiting case of an incompressible composite is not obtained by a continuous transition from the compressible; i.e., a singularity again occurs in the problem formulation. This aspect of the problem has recently been noticed by Shaffer [3] in formulating the displacement equation of equilibrium for generalized plane strain of orthotropic tubes.

This paper extends the earlier work of Herrmann [2] to orthotropic thermoelastic solids. After establishing some preliminary notation, constitutive equations valid for compressible or incompressible solids are written; in order to effect these relations it is necessary to retain an added mechanical dependent variable, and an additional constraint condition. Having established the set of field equations appropriate to both compressible and incompressible linear, orthotropic thermoelastic solids a mixed variational principle based on the Hellinger–Reissner Theorem is stated. The Euler equations of this principle are the same field equations and the natural boundary conditions are the appropriate conditions to be satisfied by the surface traction and displacement vectors. From this point on the application of the variational principle in the construction of finite element computer algorithms for the solution of boundary value problems is well-known [4].

PRELIMINARIES

The mechanical state in a linear, orthotropic thermoelastic solid is conveniently described by the (symmetric) stress and strain tensors τ_{ij} and ε_{ij}, respectively, and the displacement vector u_i†. For quasi-static problems the fifteen functions (components) are found by requiring satisfaction of: the stress equations of equilibrium,

$$\tau_{ij,j}+f_i = 0; \tag{1}$$

strain–displacement equations

$$2\varepsilon_{ij} = u_{i,j}+u_{j,i}; \tag{2}$$

and the constitutive equations

$$\varepsilon_{ij} = S_{ijkl}\tau_{kl}+\alpha_{ij}T. \tag{3}$$

In the preceding f_i is the body force vector, S_{ijkl} the elastic compliance tensor, α_{ij} the thermal expansion tensor and T the temperature change from a reference state.

For a properly posed boundary value problem there must be appended to these fifteen equations prescribed values of the displacement or traction vector on the boundary of the solid. For a compressible solid there is no formal difficulty in eliminating the strains from (2) and (3) and substituting in (1) to obtain displacement equations of equilibrium. Alternatively, the same result can be obtained by forming the strain energy density, inserting in the minimum potential energy functional and applying the variational operator. It is precisely at this point (obtaining stress in terms of strain in either case) that the procedure fails for incompressible solids. Consequently, it is necessary to modify the constitutive equation in such a manner that inversion of the strain–stress equation is always possible, irrespective of the compressibility of the solid. This is accomplished in the next section through the introduction of an additional state variable.

† State variables are referred to a fixed rectangular cartesian reference frame; the usual index notation and summation convention is inferred.

CONSTITUTIVE EQUATIONS FOR ORTHOTROPIC SOLIDS

In the sequel, anticipating applications to computer-oriented algorithms, it will be convenient to employ so-called reduced notation for the stress and strain tensors, i.e.,

$$\begin{aligned} &\sigma_1 = \tau_{11}, \quad \sigma_2 = \tau_{22}, \quad \text{etc.} \\ &\varepsilon_1 = \varepsilon_{11}, \quad \varepsilon_2 = \varepsilon_{22}, \quad \gamma_{12} = 2\varepsilon_{12}, \quad \text{etc.} \end{aligned} \tag{4}$$

With this notation the stress and strain tensors can be represented as vectors and the compliance tensor as a two-dimensional array. However, care must be exercised in transforming these quantities to other coordinate systems. For further convenience in subsequent use in the variational theorem the stress and strain "vectors" are defined as

$$\sigma_i = (\sigma_1, \sigma_2, \sigma_3, \tau_{12}, \tau_{23}, \tau_{31}) \tag{5}$$

$$\varepsilon_i = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_{12}, \gamma_{23}, \gamma_{31}). \tag{6}$$

The linear thermal expansion tensor is likewise written

$$\alpha_i = (\alpha_1, \alpha_2, \alpha_3, 0, 0, 0) \tag{7}$$

and the elastic compliance tensor is

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \tag{8}$$

with this notation the constitutive equation takes the form†

$$\varepsilon_i = S_{ij}\sigma_j + \alpha_i T \qquad i, j = 1, 2, \ldots, 6 \tag{9}$$

and the dilatation may be written as

$$\vartheta = F_i \varepsilon_i \tag{10}$$

where

$$F_i = (1, 1, 1, 0, 0, 0). \tag{11}$$

Substituting (9) into (10):

$$\vartheta = F_i S_{ij} \sigma_j + F_i \alpha_i T \tag{12}$$

$$= A_j \sigma_j + F_i \alpha_i T \tag{13}$$

where

$$A_j = F_i S_{ij} = (A_1, A_2, A_3, 0, 0, 0). \tag{14}$$

† It is assumed that the elastic axes of the orthotropic solid coincide with the fixed reference frame. In the sequel, where reduced variables appear, summation is extended over the range 1, 2, . . . , 6 unless otherwise stated.

Anticipating the need to invert (9) we form the determinant of S_{ij} as follows:

$$|S_{ij}| = (A_i\lambda_i)S_{44}S_{55}S_{66} \tag{15}$$

where

$$\begin{aligned} 3\lambda_1 &= S_{22}S_{33}-S_{23}^2+S_{23}(S_{12}+S_{13})-S_{12}S_{33}-S_{13}S_{22} \\ 3\lambda_2 &= S_{33}S_{11}-S_{13}^2+S_{13}(S_{12}+S_{23})-S_{12}S_{33}-S_{23}S_{11} \\ 3\lambda_3 &= S_{11}S_{22}-S_{12}^2+S_{12}(S_{13}+S_{23})-S_{13}S_{22}-S_{23}S_{11}. \end{aligned} \tag{16}$$

For a compressible elastic solid, the strain energy density must be positive definite; this requires that in addition to the non-vanishing of the determinant of S_{ij}, the principal minors $T_{(ii)}$ and the diagonal elements $S_{(ii)}$ of the determinant must be greater than zero [5], i.e.,

$$|S_{ij}| > 0, \qquad T_{(ii)} > 0, \qquad S_{(ii)} > 0, \qquad \text{no sum on } i. \tag{17}$$

In [3] Shaffer has shown that for a (mechanically) incompressible solid,

$$A_1 = A_2 = A_3 = 0. \tag{18}$$

These three equations (18) place restrictions on the cross-compliances of the solid, effectively reducing the number of independent elastic compliances and generalizing the result $\nu = 0{\cdot}5$ for an isotropic solid.

Since $A_i = 0$, $i = 1, 2, 3$, for an incompressible orthotropic solid, from (15) it is seen that $|S_{ij}|$ vanishes, which establishes the connection between mechanical incompressibility and vanishing of the determinant of the compliance matrix. In the sequel in dealing with an incompressible elastic solid we shall assume that (17) is replaced by the condition

$$|S_{ij}| \geq 0, \qquad T_{(ii)} > 0, \qquad S_{(ii)} > 0. \tag{19}$$

Thus, for solids that are incompressible or nearly incompressible the solution of (9) for σ_j is either not possible or numerically very sensitive. Following [1] and [2] it is desirable to modify both the stress vector σ_i and the compliance matrix S_{ij} so that (9) can be recast in a form invertible for both compressible and incompressible solids. This is accomplished by defining an additional constitutive variable and splitting the compliance matrix into two parts. Let

$$\sigma_i = HF_i + \sigma_i^* \tag{20}$$

where H is a scalar variable with the dimensions of stress and σ_i^* is the difference between the stress vector and H. (For *isotropic* solids σ_i^* is the deviator stress). Further, set

$$S_{ij} = B_{ij} + \beta_{ij} \tag{21}$$

where β_{ij} is for the present an arbitrary matrix and B_{ij} is the resulting modified compliance matrix. Equation (21) is defined to be symmetric in i and j. Substituting (20) and (21) into (9) gives

$$\varepsilon_i = (B_{ij}F_j + \beta_{ij}F_j)H + B_{ij}\sigma_j^* + \beta_{ij}\sigma_j^* + \alpha_i T. \tag{22}$$

In order to solve this equation for σ_j^* set

$$\beta_{ij}\sigma_j^* = 0. \tag{23}$$

This implies that

$$|\beta_{ij}| = 0 \tag{24}$$

Using (23), solving for σ_j^* in (22) and substituting the result into (20):

$$\sigma_j = B_{ji}^{-1}(\varepsilon_i - \alpha_i T - \beta_{ik} F_k H) \tag{25}$$

where B_{ji}^{-1}, the inverse of B_{ji}, is temporarily assumed to exist. Since H has been introduced in the constitutive equation as an additional variable, the dilatation equation (10) is retained as an independent equation. Substituting (21) and (25) in (9) and the result in (10) leads to

$$(F_i\beta_{ij}F_j + F_i\beta_{ik}B_{kl}^{-1}\beta_{lj}F_j)H - F_i\beta_{ik}B_{kj}^{-1}(\varepsilon_j - \alpha_j T) = 0. \tag{26}$$

Equations (25) and (26) comprise the constitutive equations for incompressible and nearly incompressible orthotropic solids.

We now take up the question of the existence of the inverse of the modified compliance matrix, B_{ij}. Since S_{ij} is in diagonal form for $i, j > 3$, without loss of generality we set β_{ij} zero for $i, j > 3$. Thus in considering the inverse of B_{ij} we need consider only the upper 3×3 submatrix.

To satisfy (24) set

$$\beta_{ij} = \begin{bmatrix} \beta_{11} & \sqrt{(\beta_{11}\beta_{22})} & \sqrt{(\beta_{11}\beta_{33})} \\ & \beta_{22} & \sqrt{(\beta_{22}\beta_{33})} \\ \text{symmetric} & & \beta_{33} \end{bmatrix} \tag{27}$$

Next select the β_{ij} in such a way that B_{ij} is reduced to diagonal form. This is accomplished by taking

$$\sqrt{(\beta_{(ii)}\beta_{(jj)})} = S_{ij} \quad i, j = 1, 2, 3 \quad i \neq j, \quad \text{no sum} \tag{28}$$

From (28) it follows that

$$\beta_{ii} = \frac{S_{ij}S_{ik}}{S_{jk}} \quad \text{no sum}: \quad i, j, k = 1, 2, 3 \quad i \neq j \neq k. \tag{29}$$

Substituting (29) into (21) the modified compliance matrix can now be written

$$B_{ij} = \begin{bmatrix} \dfrac{T_{23}}{S_{23}} & 0 & 0 \\ & -\dfrac{T_{13}}{S_{13}} & 0 \\ & & \dfrac{T_{12}}{S_{12}} \end{bmatrix} \tag{30}$$

where T_{ij} are the primary minors of S_{ij}. Furthermore, in the limiting case of incompressibility the vanishing of the determinant of S_{ij} implies that the primary minors are all numerically equal [6]. In the present case

$$T_{13} = -T_{12} = -T_{23} = T_{11} \tag{31}$$

where from (19) T_{11} is greater than zero. Finally it follows from (30) and (31) that the inverse of the modified compliance matrix can be written

$$B_{ij}^{-1} = -\frac{1}{T_{11}}\begin{bmatrix} S_{23} & 0 & 0 \\ 0 & S_{13} & 0 \\ 0 & 0 & S_{12} \end{bmatrix}. \tag{32}$$

Equation (32) establishes the existence of B_{ij}^{-1} in the incompressible case.

We now return to the general formulation for both compressible and incompressible solids, specializing the results for the case of isotropy. Equation (8) now takes the form

$$S_{ij} = \frac{1}{2\mu}\begin{bmatrix} \frac{1}{1+\nu} & -\frac{\nu}{1+\nu} & -\frac{\nu}{1+\nu} & 0 & 0 & 0 \\ & \frac{1}{1+\nu} & -\frac{\nu}{1+\nu} & 0 & 0 & 0 \\ & & \frac{1}{1+\nu} & 0 & 0 & 0 \\ \text{symmetric} & & & 2 & 0 & 0 \\ & & & & 2 & 0 \\ & & & & & 2 \end{bmatrix}$$

where ν, μ are Poisson's ratio and shear modulus, respectively. Thus

$$\beta_{ij} = -\frac{\nu}{2\mu(1+\nu)}F_i F_j$$

and

$$B_{ij}^{-1} = (\mu + F_{(i)}F_{(j)}\mu)\delta_{(ij)} \quad \text{no sum}$$

From these results it is easily shown that (25) and (26) reduce to

$$\sigma_i = \mu[\varepsilon_i + F_{(i)}\varepsilon_{(i)} - 2F_i\alpha T] + \frac{3\nu H}{(1+\nu)}F_i \tag{33}$$

and

$$2\mu(\vartheta - 3\alpha T) - \frac{3(1-2\nu)}{(1+\nu)}H = 0 \tag{34}$$

which apart from a constant multiplying H have been previously given in [1], [2].

A VARIATIONAL THEOREM

Having recast the constitutive equation into a form valid for both compressible and incompressible orthotropic elastic solids, i.e. (25), it is possible to return to the equilibrium equations (1) and strain–displacement equations (2) and obtain the equations of equilibrium in terms of displacements and the H variable. These equations, along with the constraint condition (26) and suitable boundary conditions, define a boundary value problem. Alternatively, the boundary value problem can be defined by a variational principle whose Euler equations and natural boundary conditions are the equilibrium equations, dilatation condition and boundary conditions respectively. The variational principle for the present case as well as the previously obtained result for isotropic materials [2], is a special case of the Hellinger–Reissner Theorem, the functional of which can be written

$$J\{\tau_{ij}, u_i\} = \int_B [W(\tau_{ij}) - \tau_{ij}\varepsilon_{ij} + f_i u_i]\,\mathrm{d}v + \int_{S_\tau} \bar{t}_i u_i\,\mathrm{d}s + \int_{S_u} t_i(u_i - \bar{u}_i)\,\mathrm{d}s \tag{35}$$

In (35) $W(\tau_{ij})$ is the complementary energy density, $\bar{t}_i$ is the surface traction vector prescribed over the part of the surface S_τ, $\bar{u}_i$ is the displacement vector prescribed over the part of the surface S_u and the strain–displacement equations are assumed to be satisfied. The mechanical state that satisfies the stress equations of equilibrium and the strain–stress equations is given by

$$\delta J = 0 \tag{36}$$

where τ_{ij} and u_i are varied independently. The state variables τ_{ij} and u_i are assumed to be of class $C^{(1)}$ and $C^{(2)}$ respectively.† In the present context the functional in (35) is modified as follows: the stress–strain relations are assumed to be satisfied, excepting the variable H, and the displacement vector u_i meets the prescribed boundary conditions on S_u. Accordingly, the functional can be expressed in terms of H and u_i and the surface integral over S_u vanishes. To facilitate writing the functional in (35) in terms of reduced variables it is necessary to introduce a set of reduced strain–displacement equations. Accordingly, we define a matrix operator D_{ij} through

$$\varepsilon_i = D_{ij}u_j \qquad i = 1, 2, \ldots, 6; \quad j = 1, 2, 3 \tag{37}$$

where

$$D_{ij} = \begin{bmatrix} \dfrac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \dfrac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial x_3} \\ \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1} & 0 \\ 0 & \dfrac{\partial}{\partial x_3} & \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} & 0 & \dfrac{\partial}{\partial x_1} \end{bmatrix} \tag{38}$$

† When the variational principle is utilized in connection with the finite element method, weaker restrictions on the state variables may be allowed. In this connection see [2].

Returning to (35), noting that

$$W(\tau_{ij})-\tau_{ij}\varepsilon_{ij} = \tfrac{1}{2}S_{ij}\sigma_i\sigma_j+\sigma_i\alpha_i T-\sigma_i\varepsilon_i,$$

we can express the functional in terms of H, u_i using (25) and (37):

$$J\{H,u_i\} = -\int_B \{B_{ij}^{-1}[\tfrac{1}{2}(D_{im}u_m)(D_{jn}u_n)-\alpha_i T D_{jn}u_n-\beta_{ik}F_kHD_{jn}u_n+\beta_{ik}F_kHT\alpha_j$$

$$+\tfrac{1}{2}\beta_{ik}\beta_{jl}F_kF_lH^2]+\tfrac{1}{2}\beta_{ij}F_iF_jH^2-f_iu_i\}\,\mathrm{d}v+\int_{S_\tau}\bar{t}_iu_i\,\mathrm{d}s. \quad (39)$$

Substituting (39) into (36) and executing the variation, (using the symmetry of B_{ij}^{-1})

$$-\int_B \{[B_{ij}^{-1}(-\beta_{ik}F_kD_{jn}u_n+\beta_{ik}F_k\alpha_jT+F_k\beta_{ki}\beta_{lj}F_lH)+F_i\beta_{ij}F_jH]\,\delta H$$

$$+[B_{ij}^{-1}(D_{im}u_m-\alpha_iT-\beta_{ik}F_kH)D_{jn}\delta u_n]-f_n\delta u_n\}\,\mathrm{d}v \quad (40)$$

$$+\int_{S_\tau}\bar{t}_n\delta u_n\,\mathrm{d}s = 0 \qquad i,j = 1,2,\ldots,6;\quad m,n = 1,2,3.$$

In order to simplify the term in the second square bracket note that

$$B_{ij}^{-1}(D_{im}u_m-\alpha_iT-\beta_{ik}F_kH) = \sigma_j. \quad (41)$$

This expression can be placed in a form suitable for application of the Divergence Theorem by using the identity

$$\sigma_jD_{jn}\delta u_n = [\tau_{mn}\delta u_n]_{,m}-\tau_{mn,m}\delta u_n \quad (42)$$

where τ_{mn} is the symmetric stress tensor

$$\tau_{mn} = \begin{bmatrix}\sigma_1 & \sigma_4 & \sigma_6\\ \sigma_4 & \sigma_2 & \sigma_5\\ \sigma_6 & \sigma_5 & \sigma_3\end{bmatrix} \quad (43)$$

Accordingly, using (41), (42) in the second square bracket of (40) and applying the Divergence Theorem leads to

$$-\int_B[\sigma_jD_{jn}\delta u_n-f_n\delta u_n]\,\mathrm{d}v = -\int_{S_\tau}\tau_{mn}\nu_m\delta u_n\,\mathrm{d}s+\int_B(\tau_{mn,m}+f_n)\delta u_n\,\mathrm{d}v \quad (44)$$

Using this result (40) can be written

$$\int_B\{[\text{Eq. (26)}]\delta H+[\tau_{mn,m}+f_n]\delta u_n\}\,\mathrm{d}v+\int_{S_\tau}(\bar{t}_n-\tau_{mn}\nu_m)\delta u_n\,\mathrm{d}s = 0. \quad (45)$$

Appealing to the usual lemma of the calculus of variations, the independent vanishing of the bracketed expressions multiplying δH and δu_n is equivalent to the dilatation condition (26) and the stress equations of equilibrium (or displacement equations of equilibrium if (41), (43) are used). Furthermore vanishing of the surface integral is equivalent to satisfaction of the traction boundary condition. In the special case of isotropy (45) reduces to the result obtained in [2].

REFERENCES

[1] L. R. HERRMANN and R. M. TOMS, A reformulation of the elastic field equations in terms of displacements, valid for all admissible values of Poisson's ratio. *J. appl. Mech., Trans. ASME* **86**, 140–141 (1964).
[2] L. R. HERRMANN, Elasticity equations for incompressible and nearly incompressible materials by a variational theorem. *AIAA Jnl* **3**, 1896–1900 (1965).
[3] B. W. SHAFFER, Generalized plain strain of pressurized orthotropic tubes. *J. Engng Ind., Trans. ASME* **87**, 337–343 (1965).
[4] O. C. ZIENKIEWICZ and G. S. HOLISTER, *Stress Analysis*, Chapters 7, 8, 9. Wiley (1965).
[5] W. L. FERRAR, *Algebra*, p. 138. Oxford University Press (1941).
[6] THOMAS MUIR, *A Treatise on the Theory of Determinants*, p. 372. Longmans, Green (1933).

(*Received* 8 *January* 1968; *revised* 11 *March* 1968)

Абстракт—Приводится смешаную вариационную теорему для линейного ортотропного термоупругого твердого тела. Переменные механического состояния являются переменными вектора перемещений и скаляра напряжений. Уравнения Эйлера для вариационного принципа оказываются уравнениями перемещения равновесия и условием отношения переменной напряжения к изменению деформации и температуры. Важной особенностью принципа является факт, что уравнения поля так для сжимаемых, как и для несжимаемых тел, можно обобщить. В связи с применениями разработки алгорифмов конечного элемента для вычислительных машин, с целью получения решения краевых задач, определяется надлежащими образом обусловенная система уравнений для почти несжимаемых тел.